

# Bulk–Boundary Correspondence for Chiral Symmetric Quantum Walks

János K. Asbóth<sup>1</sup> and Hideaki Obuse<sup>2</sup>

<sup>1</sup> *Institute for Solid State Physics and Optics, Wigner Research Centre,  
Hungarian Academy of Sciences, H-1525 Budapest P.O. Box 49, Hungary*

<sup>2</sup> *Department of Applied Physics, Hokkaido University, Sapporo 060-8628, Japan*  
(Dated: Winter 2013)

A discrete-time quantum walk (DTQW) is defined by a periodic sequence of operations on a quantum particle. According to the choice of the starting time of the period, different effective Hamiltonians can be associated to the same quantum walk. We define a DTQW to have chiral symmetry (CS) if at least one of these Hamiltonians has CS. This can be ensured by using an “inversion symmetric” pulse sequence, which automatically gives two, inequivalent effective Hamiltonians with CS. We show that the sum and difference of the associated winding numbers, divided by two, gives the bulk topological invariants of a DTQW with CS, which control the number of 0 and  $\pi$  energy edge states. We illustrate this bulk–boundary correspondence for the DTQW on the example of the “4-step quantum walk”, where tuning CS and particle-hole symmetry realizes edge states in various symmetry classes.

PACS numbers: 05.30.Rt, 03.67.-a, 03.65.Vf

The realization that band insulators can have non-trivial topological properties which determine the low-energy physics at their boundary has been a rich source of new physics in the last decade. The general theory of topological insulators and superconductors [1, 2] classifies gapped Hamiltonians according to their dimension and their symmetries [3]. As very few real-life materials are topological insulators, there is a strong push to develop model systems, “artificial materials”, that simulate topological phases [4].

Discrete-time quantum walks (DTQW)[5–8] offer a versatile way of simulating topological insulators[9–11]. Beyond that, they also have unique topological features, with no counterpart in standard solid-state setups. In DTQWs with particle–hole symmetry (PHS), edge states, “Majorana modes” can have two protected quasienergies:  $\varepsilon = 0$  or  $\pi$  (time is measured in units of the timestep and  $\hbar = 1$ ). Generalizing the results for periodically driven systems [12], a pair of topological invariants ( $\mathbb{Z}_2 \times \mathbb{Z}_2$ ) has been defined, which control the appearance of these topologically protected bound states [13]. Both 0 and  $\pi$  energy Majorana edge states have been experimentally observed in an optical realization of a quantum walk [14].

As for chiral symmetry (CS), the situation of DTQW’s is much less clear. Even for the simplest 1-dimensional DTQW, it is disputed whether it even has CS [9] or not [13]. Although it is expected that CS should imply a  $\mathbb{Z} \times \mathbb{Z}$  bulk topological invariant, this has not yet been found for DTQW’s. As opposed to the case of PHS, there is also not much to draw on from periodically driven systems. What DTQW’s have CS? How can the bulk “winding number” be expressed for DTQW’s with CS? These are the problems we tackle in this paper.

A DTQW concerns the dynamics of a particle, “walker”, whose wavefunction is given by a vector,  $|\Psi\rangle =$

$\sum_{x=0}^N \sum_{s=-1,1} \Psi(x, s)|x, s\rangle$ . Here,  $x = 1, \dots, N$  is the discrete position, and  $s = \pm 1$  indexes the two orthogonal internal states of the walker, the “coin eigenstates”, which we also refer to as “spin”. The dynamics, instead of given by a time-independent Hamiltonian, is realized using a periodic sequence of alternating “step” and “coin rotation” operations.

The step operations are translations of the particle by one lattice site depending on the value of the “coin”, the  $z$ -component of its spin. These are described by unitary operators  $S_s$ , where  $s$  is either  $+$  or  $-$ , and

$$S_{\pm} = \sum_{x=1}^N (|x \pm 1\rangle\langle x| \otimes |\pm 1\rangle\langle \pm 1| + |x\rangle\langle x| \otimes |\mp 1\rangle\langle \mp 1|). \quad (1)$$

For simplicity, we take periodic boundary conditions.

Between each two steps, a site-dependent local “coin rotation”  $R_j$  on the walker’s internal state is performed. We consider

$$R_j = \sum_{x=1}^N |x\rangle\langle x| \otimes R(\chi_j(x), \theta_j(x)); \quad (2)$$

$$R(\chi, \theta) = \begin{pmatrix} \cos \theta - i \sin \theta \sin \chi & -\sin \theta \cos \chi \\ \sin \theta \cos \chi & \cos \theta + i \sin \theta \sin \chi \end{pmatrix} \quad (3)$$

$$= \exp[-i\theta(\cos(\chi)\sigma_y + \sin(\chi)\sigma_z)] \quad (4)$$

This allows breaking PHS via the angle  $\chi$  [9]. Details of how the local operations  $R_j$  are performed do not influence the DTQW, all the information about them is summarized in the corresponding unitaries  $R_j$ .

One period of the DTQW is defined by  $|\Psi(t+1)\rangle = U_0|\Psi(t)\rangle$ , for  $t \in \mathbb{Z}$ . Here the unitary timestep (Floquet) operator is composed of  $2M$  successive pulses,

$$U_0 = S_M R_M S_{M-1} R_{M-1} \dots S_1 R_1. \quad (5)$$

A period has to include an equal number of  $S_+$  and  $S_-$  pulses, otherwise timestep operator has quasienergy

winding [15], and cannot have gaps. Thus,  $M$  is even. We take each pulse to have a duration  $1/2M$ , without losing generality. We can also shift the starting time of the period by  $T$ , giving the dynamics as  $|\Psi(t+1+T)\rangle = U_T|\Psi(t+T)\rangle$ , for any  $t \in \mathbb{Z}$ . The starting time  $T$  has to be during a rotation, restricting it to  $T = (l-1)/M + y/(2M)$ , for  $1 \leq l < M$ , and  $0 \leq y < 1$ . The shifted Floquet operator reads

$$U_T = R_{M+l}^y S_{M+l-1} R_{M+l-1} \dots S_{l+1} R_{l+1} S_l R_l^{1-y}, \quad (6)$$

where  $R_j^y = \sum_x |x\rangle\langle x| \otimes R(\chi_j(x), y\theta_j(x))$ . This is a unitary transform of  $U_0$ , we refer to the transformation connecting the  $U_T$  to  $U_0$  as a “gauge” transformation. We don’t consider splitting translation operators  $S$ . In many realizations of DTQWs, e.g. the ones on atoms trapped in optical lattices or on photons on an optical table, performing only part of a shift operation would leave the walker between sites, and its description would necessitate an increased Hilbert space.

A DTQW can be seen as a stroboscopic simulator of an effective Hamiltonian  $H_{\text{eff},T}$ , which is associated to the Floquet operator by

$$U_T \equiv e^{-iH_{\text{eff},T}\tau}. \quad (7)$$

The effective Hamiltonian is uniquely defined by this equation if we restrict its eigenvalues, the quasienergies, to an “energy Brillouin zone”,  $-\pi < \varepsilon \leq \pi$ . This is completely analogous to the restriction of the quasimomentum to the first Brillouin zone.

Previous work on CS in DTQWs has focussed on whether  $H_{\text{eff},0}$  has CS, and identifying the associated topological invariant. We realize that it is important to widen the scope: a DTQW has CS, *if there is a gauge where its effective Hamiltonian has CS*: If a time  $T$  and a unitary operator  $\Gamma$  acting on the coin space can be found, with  $\Gamma^2 = 1$ , that  $\Gamma U_T \Gamma = U_T^{-1}$ . This unitary time-reversal symmetry is equivalent to CS of the effective Hamiltonian  $H_{\text{eff},T}$ .

We now proceed to derive the bulk–boundary correspondence for DTQWs with CS, i.e., a formula for the  $\mathbb{Z} \times \mathbb{Z}$  bulk topological invariant controlling the appearance of 0 and  $\pi$  energy edge states. To do this, we first show that DTQW’s with CS have not only one, but two inequivalent chiral symmetric gauges.

Based on the definition of CS above, a DTQW has CS represented by  $\Gamma = \sigma_x$ , if the sequence of operations defining the walk has an “inversion point”. By this, we mean that there is an  $l$ , with which for every  $j$ :

$$R_{l-j} = R_{l+j}; \quad (8)$$

$$S_{l-j} = S_+ \leftrightarrow S_{l+j+1} = S_-. \quad (9)$$

As we show below, this not only ensures that there is a gauge where the effective Hamiltonian has CS, but two inequivalent such gauges.

To prove that “inversion symmetry” of the pulse sequence gives two CS gauges, we begin by noticing that

$$\sigma_x S_- \sigma_x = S_+^{-1}, \quad (10)$$

whereby also  $\sigma_x S_+ \sigma_x = S_-^{-1}$ . Furthermore, since the local unitaries are rotations  $R_j$  about axes that have no  $x$ -component,

$$\sigma_x R(\theta, \chi) \sigma_x = R(\theta, \chi)^{-1} = R(-\theta, \chi). \quad (11)$$

Take the sequences of  $M$  operations just after and just before the middle of the “inversion point”,

$$F = R_{l+M/2}^{1/2} S_{l-1+M/2} R_{l-1+M/2} \dots R_{l+1} S_l R_l^{1/2}; \quad (12)$$

$$G = R_l^{1/2} S_{l-1} R_{l-1} \dots R_{l+1-M/2} S_{l-M/2} R_{l-M/2}^{1/2}. \quad (13)$$

These give us two Floquet operators for the walk:

$$U' = FG; \quad U'' = GF. \quad (14)$$

Using relations (10) and (11), we have that time reversal can be done *during* a period,  $\Gamma F \Gamma G = 1$ , whereby  $G = \Gamma F^{-1} \Gamma$ . From this it is straightforward to show that both  $U'$  and  $U''$  are chiral symmetric.

CS allows a definition of sublattices, via the projection operators  $\Pi_A = (1 + \Gamma)/2$ ,  $\Pi_B = (1 - \Gamma)/2$ . Eigenstates of  $H'_{\text{eff}}$  with quasienergy  $\varepsilon \neq 0, \pi$  can be chosen to have equal support on both sublattices. Stationary states with quasienergies 0 or  $\pi$ , however, can be chosen to be on a single sublattice in the chiral gauge  $U'$ , i.e., their wavefunctions in this gauge are eigenstates of  $\Gamma$ .

For the bulk–boundary correspondence, we consider an inhomogeneous DTQW with CS, consisting of a translationally invariant “ $L$ ” bulk at  $1 \ll x \ll d$  and an “ $R$ ” bulk at  $d \ll x \ll N$ . There are (smooth or sharp) boundaries between the two bulks around  $x \approx d$  and  $x \approx 1$ . In the chiral symmetric gauge  $U'$ , the two bulks have effective Hamiltonians  $H'_{\text{eff},L}$  and  $H'_{\text{eff},R}$ . We assume both bulk Hamiltonians have gaps around  $\varepsilon = 0$  and  $\varepsilon = \pi$ . Therefore, stationary states with quasienergies  $\varepsilon = 0$  or  $\pi$  must have wavefunctions confined to the edges, exponentially decaying towards the bulks. The edge around  $x \approx d$  hosts  $m'_A$  ( $m'_B$ ) edge states on sublattice  $A$  ( $B$ ), given by

$$m'_A = m'_{A,0} + m'_{A,\pi}; \quad m'_B = m'_{B,0} + m'_{B,\pi}, \quad (15)$$

where the second index stands for the energy. We are looking for the topological invariants of the bulk parts of the walk,  $\nu_{X,\varepsilon}$ , with  $X = L, R$  and  $\varepsilon = 0, \pi$ , whose differences give us the number of topologically protected edge states separately for each energy,

$$\nu_{L,\varepsilon} - \nu_{R,\varepsilon} = m'_{A,\varepsilon} - m'_{B,\varepsilon}. \quad (16)$$

The first step towards the topological invariants is the standard winding number  $\nu'_X$  [3] associated to the bulk

effective Hamiltonian  $H'_{\text{eff},X}$ , where  $X = L$  or  $X = R$  in gauge  $U'$ . This is obtained from the bulk, translational invariant part of the Floquet operator, diagonal in momentum space:  $U'_X = \sum_k |k\rangle\langle k| \otimes U'_X(k)$ , and  $U'_X(k) = e^{-iH_{\text{eff},X}(k)}$ . Instead of the effective Hamiltonian, it is convenient to calculate with  $H'_X(k) = \sin[H_{\text{eff},X}(k)]$ . This has the same CS and the same winding number as  $H_{\text{eff},X}(k)$  (and as its flattened version  $Q = \text{sgn}[H_{\text{eff},X}(k)]$  [3]), but can be obtained much more efficiently via  $H'_X(k) = [U'_X(k)^\dagger - U'_X(k)]/(2i)$ . In a basis where  $\Gamma = \text{diag}(1, \dots, 1, -1, \dots, -1)$  is a diagonal matrix with an equal number of  $+1$  and  $-1$  elements, the matrix of  $H'_X(k)$  is block off diagonal because of CS. We name its upper right block  $h'_X(k)$ . The winding number  $\nu'_X$  reads

$$\nu'_X = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \frac{d}{dk} \log \det h'_X(k). \quad (17)$$

The winding number  $\nu'_X$  is related to the difference of the bulk polarizations on the two sublattices in bulk  $X$  [16]. Therefore, it cannot differentiate between 0 and  $\pi$  energy edge states, and can only be used to obtain the sum of all topologically protected edge states around  $x \approx d$ :

$$\nu'_L - \nu'_R = m'_{A,0} + m'_{A,\pi} - m'_{B,0} - m'_{B,\pi}. \quad (18)$$

However, there is the other CS gauge,  $U''$ , where we have

$$\nu''_L - \nu''_R = m''_{A,0} + m''_{A,\pi} - m''_{B,0} - m''_{B,\pi}. \quad (19)$$

We need to combine the information from the two CS gauges to obtain the topological invariants.

We can obtain a simple connection between the two CS gauges by considering an edge state. In the gauge of  $U' = FG$ , the edge state has a wavefunction  $\Psi$ , entirely on sublattice  $A$  (or  $B$ ), i.e.,  $\Gamma\Psi = (-1)^g\Psi$ , with  $g = 0$  (or  $1$ ). In other gauges, where  $U_T$  has no CS, the energy of the edge state has to remain the same, but its wavefunction can extend over both sublattices. In the other CS gauge  $U'' = GF$ , however, its wavefunction,  $\Phi = G\Psi$ , again has to be entirely on a single sublattice. This can be  $A$  (or  $B$ ), whereby  $\Gamma\Phi = (-1)^f\Phi$ , with  $f = 0$  (or  $1$ ). Consider  $GF\Phi = G\Gamma G^{-1}\Gamma\Phi = G\Gamma G^{-1}(-1)^f\Phi = G\Gamma(-1)^f\Psi = (-1)^{g+f}G\Psi = (-1)^{g+f}\Phi$ . This shows that 0 ( $\pi$ ) energy edge states are on the same (opposite) sublattice in the two CS gauges. This can be summarized as

$$m''_A = m'_{A,0} + m'_{B,\pi}; \quad m''_B = m'_{B,0} + m'_{A,\pi}. \quad (20)$$

To obtain the number of protected edge states at zero and  $\pi$  energies separately, we simply substitute Eqs. (20) into Eqs. (18), (15) and (19), and rearrange to obtain

$$m'_{A,0} - m'_{B,0} = \frac{\nu'_L + \nu''_L}{2} - \frac{\nu'_R + \nu''_R}{2}; \quad (21)$$

$$m'_{A,\pi} - m'_{B,\pi} = \frac{\nu'_L - \nu''_L}{2} - \frac{\nu'_R - \nu''_R}{2}. \quad (22)$$

We compare this with Eq. (16), and simply read off the bulk topological invariants  $(\nu_0, \nu_\pi)$  as

$$(\nu_0, \nu_\pi) = \left( \frac{\nu' + \nu''}{2}, \frac{\nu' - \nu''}{2} \right). \quad (23)$$

This, the bulk-edge correspondence for DTQW's with CS, is the main result of the paper.

Having derived a general formula for the topological invariant of 1D DTQWs with CS, we now discuss an example where the differences between CS and PHS come into play. To arrive to the example, first consider the “split-step walk” of Kitagawa et al.[9], given by  $U_0 = S_+R(0, \phi)S_-R(0, \theta_1)$ . There, both PHS and CS are present, and we find that the  $\nu_\varepsilon$  are one-to-one functions of the invariants  $Q_\varepsilon$  induced by PHS [13]:  $\nu_\varepsilon = 1/2 - Q_\varepsilon$ , for both  $\varepsilon = 0, \pi$ . (An interesting special case is the simple quantum walk, obtained by setting  $\phi = 0$ .) We can break PHS by using nonzero angles  $\chi$ . To be able to break CS, we can consider a longer period of pulses, a “4-step DTQW”, given by

$$U_0 = S_+R_4S_+R_3S_-R_2S_-R_1. \quad (24)$$

This walk has no CS if  $R_2 \neq R_4$ , but has CS if  $R_2 = R_4$ , with  $F = R_3^{1/2}S_-R_2S_-R_1^{1/2}$  and  $G = R_1^{1/2}S_+R_4S_+R_3^{1/2}$ . The 4-step walk also has the advantage that the effective Hamiltonian will have longer range hoppings, and thus we can expect higher values of the winding numbers. This is entirely analogous to adding a 3rd nearest neighbor hopping term to the SSH model.

The topological invariants in a section of the phase space of the 4-step DTQW with both CS and PHS (Cartan class BDI [3]) are shown in Fig. 1. Here we set all  $\chi_j = 0$  to ensure PHS,  $\theta_2 = \theta_4$  to ensure CS, and set  $\theta_1 = 0$  for simplicity. We restrict  $\theta_2$  to  $-\pi/2 < \theta_2 < \pi/2$  since adding  $\pi$  to both  $\theta_2$  and  $\theta_4$  just brings two factors of  $-1$  that cancel out in both chiral gauges. Generic values of  $\theta_2 = \theta_4$  and  $\theta_3$  give effective Hamiltonians with gaps around both  $\varepsilon = 0$  and  $\varepsilon = \pi$ . Examples for these are the points  $C(\theta_2 = \pi/20, \theta_3 = \pi/4)$ ,  $D(\theta_2 = 0, \theta_3 = -\pi/4)$ , and  $E(\theta_2 = \pi/4, \theta_3 = \pi/4)$ .

To see the effects of breaking the symmetries on edge states, we consider two inhomogeneous systems, consisting of two domains of 40 sites each, with sharp boundaries in between. The inhomogeneous rotations read

$$R_j = \sum_{x=1}^{40} |x\rangle\langle x| \otimes R_{j,X} + \sum_{x=41}^{80} |x\rangle\langle x| \otimes R_{j,C}, \quad (25)$$

where  $X = D$  or  $X = E$ , and  $C$  refer to the parameter sets of defined in the previous paragraph and indicated in Fig. 1.

We break PHS (realizing Cartan class AIII) in a controlled way by introducing a nonzero  $\chi_3$  in the bulk  $0 < x < 41$ . As long as the bulk gaps are still open, breaking PHS does not change the edge state energies,

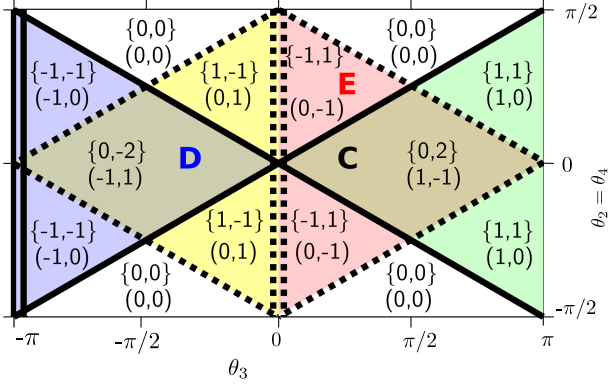


FIG. 1: (Color online) Parameter space of the 4-step DTQW with PHS ensured by  $\chi_j = 0$ , CS ensured by  $\theta_2 = \theta_4$ , and  $\theta_1$  set to 0. The DTQW has effective Hamiltonians with gaps around both  $\varepsilon = 0$  and  $\varepsilon = \pi$ , except at the gapless points where gaps close at  $\varepsilon = 0$  (solid lines) or  $\varepsilon = \pi$  (dashed lines). Single lines indicate that the gap closes at a single  $k$ , at either  $k = 0$  or  $k = \pi$ . Double lines indicate double gap closings, at  $k = \pm\pi/2$ . For each gapped domain, the corresponding pair of winding numbers  $\{\nu', \nu''\}$  as well as the pair of topological invariants  $(\nu_0, \nu_\pi)$ , cf. Eq. (23), are shown. Letters “C”, “D” and “E” indicate sets of parameters used for the inhomogeneous quantum walk, with rotation as in Eq. (25).

as shown in Fig. 2 A1), B1). The edge state energies are still protected by CS, and can only move from their original values if the bulk gap closes (at  $\chi_3 = \pi/2$  for the D-C boundary). We break CS (realizing Cartan class D), by changing  $\theta_2 - \theta_4$  in the “L” bulk. A pair of edge states on the same edge at the same energy can now break apart, becoming PHS partners of each other. This can be seen in Fig. 2 A2) at both 0 and  $\pi$  energy. However, a single edge state, as the one between bulks B and C, is still protected by PHS when CS is broken, as seen in Fig. 2 B2). Finally, to check that no extra hidden symmetries remain, we break both CS and PHS (realizing Cartan class A). In that case the edge state energies are not protected anymore, cf Fig. 2 A3), B3). This shows that our description of the relevant symmetries of the DTQW was indeed exhaustive.

To summarize, we gave a definition of CS for DTQWs, and derived the corresponding bulk topological invariants, using the fact that the walk is defined by a sequence of operations, rather than just by its unitary timestep operator. The approach presented here based on finding the “symmetric gauges” should generalize to periodically driven quantum systems [15, 17–20]. In such setups, PHS has been shown to lead to 2 types of “Floquet Majorana fermions” [12], which should have clear signals in transport [21] and can also be useful for quantum information processing [22]. Theoretical proposals have already seen several such states at a single edge if the driving also ensures CS [23]. The bulk topological invariant controlling

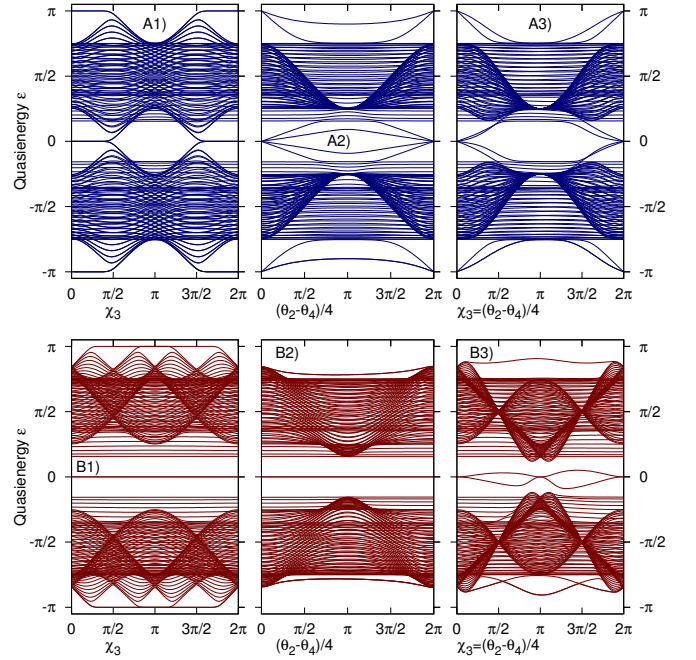


FIG. 2: (Color online) Spectra of an inhomogeneous “4-step walk” on  $N = 60$  sites as defined by Eqs. (24),(25), with two domains: D and C (top row), and E and C (bottom row). Left panels: We break PHS via  $\chi_3$  for  $x < 41$  on A1) and B1). As long as the bulk gaps are still open, the edge state energies do not change as they are still protected by CS. Middle panels, A2) and B2): we break CS by setting  $\theta_2 \rightarrow \theta_2 + \Delta\theta$ ,  $\theta_4 \rightarrow \theta_4 - \Delta\theta$  for  $n < 31$ . This lifts the degeneracy of edge states on the same edge pairwise. At the interface between E and C, the unpaired edge state remains (B2). Right panels, A3) and B3): as both PHS and CS are broken, no topologically protected edge states remain. In B3), the unpaired edge states at both edges are displaced in energy.

the number of these edge states is as yet unknown, but it could be derived using the approach of this paper.

Although 2-dimensional DTQWs have already been realized in experiments [24], their topological invariants are largely unexplored. In 2 dimensions, edge states can exist in the absence of symmetries; the related bulk–boundary correspondence for periodically drive systems has only recently been found [25]. The approach of identifying the “symmetric gauges” should be a key idea for the description of other symmetry classes for both periodically driven quantum systems and DTQWs.

JKA thanks J. Edge and A. Gábris, and HO thanks N. Kawakami, Y. Nishimura, and T. Kitagawa for helpful discussions. This research was realized in the frames of TAMOP 4.2.4. A/1-11-1-2012-0001 “National Excellence Program – Elaborating and operating an inland student and researcher personal support system”, subsidized by the European Union and co-financed by the European Social Fund. This work was also supported by the Hungarian National Office for Research and Technology under the contract ERC\_HU\_09 OPTOMECH and

the Hungarian Academy of Sciences (Lendület Program, LP2011-016).

- 
- [1] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010), URL <http://link.aps.org/doi/10.1103/RevModPhys.82.3045>.
  - [2] X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011), URL <http://link.aps.org/doi/10.1103/RevModPhys.83.1057>.
  - [3] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, *New Journal of Physics* **12**, 065010 (2010).
  - [4] P. Hauke, O. Tieleman, A. Celi, C. Ölschläger, J. Simonet, J. Struck, M. Weinberg, P. Windpassinger, K. Sengstock, M. Lewenstein, et al., *Phys. Rev. Lett.* **109**, 145301 (2012), URL <http://link.aps.org/doi/10.1103/PhysRevLett.109.145301>.
  - [5] Y. Aharonov, L. Davidovich, and N. Zagury, *Phys. Rev. A* **48**, 1687 (1993), URL <http://link.aps.org/doi/10.1103/PhysRevA.48.1687>.
  - [6] J. Kempe, *Contemporary Physics* **44**, 307 (2003), URL <http://www.tandfonline.com/doi/abs/10.1080/00107151031000110776>.
  - [7] A. Ambainis, *International Journal of Quantum Information* **01**, 507 (2003), URL <http://www.worldscientific.com/doi/abs/10.1142/S0219749903000383>.
  - [8] A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, P. J. Mosley, E. Andersson, I. Jex, and C. Silberhorn, *Phys. Rev. Lett.* **104**, 050502 (2010), URL <http://link.aps.org/doi/10.1103/PhysRevLett.104.050502>.
  - [9] T. Kitagawa, M. S. Rudner, E. Berg, and E. Demler, *Phys. Rev. A* **82**, 033429 (2010), URL <http://link.aps.org/doi/10.1103/PhysRevA.82.033429>.
  - [10] T. Kitagawa, *Quantum Information Processing* **11**, 1107 (2012).
  - [11] H. Obuse and N. Kawakami, *Phys. Rev. B* **84**, 195139 (2011), URL <http://link.aps.org/doi/10.1103/PhysRevB.84.195139>.
  - [12] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, *Phys. Rev. Lett.* **106**, 220402 (2011), URL <http://link.aps.org/doi/10.1103/PhysRevLett.106.220402>.
  - [13] J. K. Asbóth, *Phys. Rev. B* **86**, 195414 (2012), URL <http://link.aps.org/doi/10.1103/PhysRevB.86.195414>.
  - [14] T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White, *Nature Communications* **3**, 882 (2012).
  - [15] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, *Phys. Rev. B* **82**, 235114 (2010), URL <http://link.aps.org/doi/10.1103/PhysRevB.82.235114>.
  - [16] J. K. Asbóth and H. Obuse, unpublished (2013).
  - [17] L. N. H., G. Refael, and V. Galitski, *Nature Physics* **7**, 490495 (2011).
  - [18] J. Cayssol, B. Dóra, F. Simon, and R. Moessner, *Phys. Status Solidi RRL* **7**, 101 (2013).
  - [19] K. Sun, W. V. Liu, A. Hemmerich, and S. D. Sarma, *Nature Physics* **8**, 6770 (2012).
  - [20] J. P. Dahlhaus, J. M. Edge, J. Tworzydło, and C. W. J. Beenakker, *Phys. Rev. B* **84**, 115133 (2011), URL <http://link.aps.org/doi/10.1103/PhysRevB.84.115133>.
  - [21] A. Kundu and B. Seradjeh, arxiv p. arXiv:1301.4433 (2013).
  - [22] D. E. Liu, A. Levchenko, and H. U. Baranger, arxiv p. arxiv:1211.1404v2 (2012).
  - [23] Q.-J. Tong, J.-H. An, J. Gong, H.-G. Luo, and C. H. Oh, arxiv p. arxiv:1211.2498v1 (2012).
  - [24] A. Schreiber, A. Gábris, P. P. Rohde, K. Laiho, M. Stefanák, V. Potoček, C. Hamilton, I. Jex, and C. Silberhorn, *Science* **336**, 55 (2012), <http://www.sciencemag.org/content/336/6077/55.full.pdf>, URL <http://www.sciencemag.org/content/336/6077/55.abstract>.
  - [25] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, arxiv p. arxiv:1212.3324v1 (2012).